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Fresnel diffraction from sector apertures

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Abstract: The derivation of the analytic representation of the diffraction field from the sector aperture in the Fresnel zone is presented. The new solution is expressed in the form of a convergent series on incomplete cylindrical functions in the Poisson form that makes it possible to calculate and obtain three-dimensional patterns of the field amplitude in the whole image plane. The obtained patterns correspond to the Fresnel approximation and are true for arbitrary values of the opening angle and for the radius of sectoral aperture.

1. Introduction

In the classical cases of physical optics, the diffraction field represents a rapidly oscillating surface integral [1] and therefore it is very difficult for numerical evaluations, which in most cases become almost impossible-doing task.

Historically, such integral first was presented in analytical form in 1885-1886 by E Lommel for rectangular and circular apertures [2,3]. Owing to the complexity of the problem for more than 135 years after E. Lommel, neither analytical representation of physical optics integral for other configurations of apertures was proposed and the subject of continuous mathematical research was the transformation of them into linear integrals, which also have their limitations owing to the same rapid oscillations.

Since the physical optics diffraction is widely used in optics itself, acoustics, radiophysics and other branches of physics, the search for such integrals of analytical (integral-free) representations, which would lead to the formation of effective algorithms for their calculation, is relevant. This paper examines the case of Fresnel diffraction by sector aperture. The case of the sector aperture is also interesting because, owing to the analytical community, the expected problem solution can serve as a basis for formation a whole series of solutions when diffraction by other angular and segment areas where there are no analytical solutions.

2. Sector aperture

In the case of spherical wave incident from a point source to the plane of aperture with radius R and opening angle φ, it is appropriate to choose polar coordinates ρ, φ, with the center superposed with the center of sector aperture (Fig. 1). The problem is, in fact, in finding an analytic expression for the multiplier of diffraction attenuation $\Phi = W/W_0$ where W is the diffraction field from aperture, and $W_0 = \exp\left(i\frac{2\pi}{\lambda}r_0\right)/\,r_0$ is the field of direct unperturbed wave at the observation point behind aperture at distance r_0 from the source; λ is wavelength.

Desired function Φ is found as the sum (the beam passes on aperture) or the difference (the beam passes outside aperture) of the results of integration over two adjacent constituent sectors, when the beam passes on their common straight edge (Fig. 1):

$$\Phi\,(\rho_0, \phi_0) = \Phi\,(\rho_0, \phi_1) \pm \Phi\,(\rho_0, \phi_2), \quad \text{where} \tag{1}$$

$$\Phi\,(\rho_0, \phi_j) = -\frac{i}{2}\pi\, n\, \exp(i\pi\, n_0) \int\limits_0^1 \exp(i\pi\, nu^2)\, u \left[\frac{2}{\pi}\int\limits_0^{\phi_j} \exp(-i2\pi\sqrt{n\, n_0}\, u\cos\phi_j)\, d\phi\right] du \tag{2}$$

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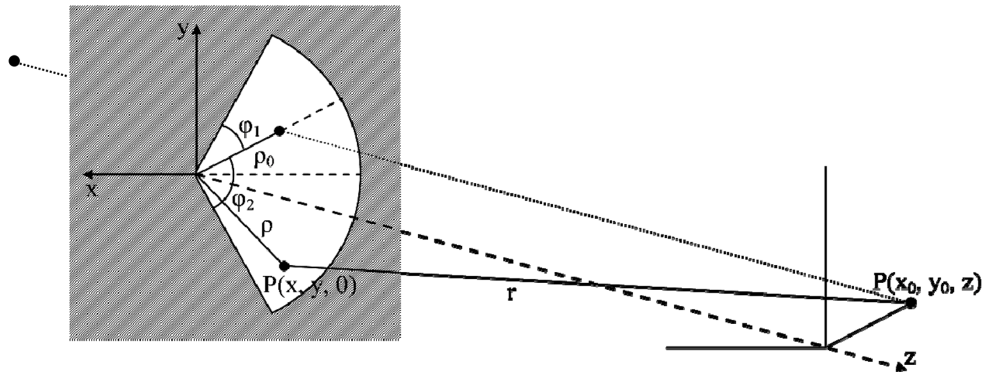

Fig. 1. Task geometry

The following designations are accepted here: $\sqrt{n} = R/b_1$, $\sqrt{n_0} = \rho_0/b_1$ are the relative magnitudes of sector radii and of the point of beam passing in Fresnel zones, where $b_1 = \sqrt{\frac{\lambda \cdot r_1(r_0-r_1)}{r_0}}$ is the radius of the first Fresnel zone in the plane of aperture, r_1 is the distance from aperture to source; r_0 is the distance from source to P $y_0(x_0, y_0$, z; u = ρ / R is the integration variable relatively to the magnitude of sector radius. Polar coordinates ρ,φ, with the center superposed with the sector aperture center are used: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $\rho_0 = \sqrt{x_0^2 + y_0^2}$.

All accepted in the task approximate conditions correspond to the typical conditions of Fresnel diffraction [1]: the normal incidence of a wave to two-dimensional object; the wavelength is small compared to the object size, and the object size and the deviation of observation point from z axis are small compared to distance r_0, that makes it possible to take a small-angle approximation. Although Eq. (2) was formulated in the nineteenth century, its analytical solution was waiting for the appearance of the necessary mathematical apparatus [4,5].

3. Diffraction field

Let us present Eq. (2) in the form of

$$\Phi(\rho_0, \phi) = -\frac{i}{2}\pi n \exp(i\pi n_0)\int_0^1 \exp(i\pi n u^2)u \cdot E_0^-(\phi, pu)\cdot du, \tag{3}$$

where $E_0^-(\phi, pu) = \frac{2}{\pi}\int_0^\phi \exp(-i\pi u\cos\phi)\cdot d\phi$ is the incomplete cylindrical function in Poisson form with index ν = 0 [4], and parameter $p = 2\pi\sqrt{n_0}\sqrt{n}$; Let us also introduce following parameter $q{=}2\pi n$, which will be needed hereinafter.

Let us apply to Eq. (3) the successive partial integration, as it was done by Lommel [2] in the case of a circular aperture. For this purpose, the following recurrent relations are used from the theory of incomplete cylindrical functions:

$$L_\nu E_\nu^-(\phi, h) = 2h^\nu \sin^{2\nu-1}\phi\left[\frac{\cos\phi}{A_\nu h} - i\frac{\sin^2\phi}{A_{\nu+1}}\right]e^{-ih\cos\phi} \tag{4}$$

$$L'_\nu E_\nu^-(\phi, h) = 2h^\nu \sin^{2\nu-1}\phi\left[\frac{\sin\phi}{A_\nu h} + i\frac{\sin^2\phi}{A_{\nu+1}}\right]e^{-ih\cos\phi} \tag{5}$$

where operators L_ν, L'_ν have the following values

$$L_\nu E_\nu^-(\phi,h) \equiv E_{\nu-1}^-(\phi,h) + E_{\nu+1}^-(\phi,h) - \frac{2\nu}{h} E_\nu^-(\phi,h) \tag{6}$$

$$L'_\nu E_\nu^-(\phi,h) \equiv E_{\nu-1}^-(\phi,h) + E_{\nu+1}^-(\phi,h) - 2\frac{d}{dh} E_\nu^-(\phi,h) \tag{7}$$

In Eqs. (4), (5) quantity $A_\nu = 2\nu\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})$, was introduced from [4], where Γ(ν) is gamma-function.

Further conversion of these relations was carried out in order to reduce them to such form, so that when partial integration of Eq. (3), the convergence series would appear for two cases: at and at $n_0 \geq n$, that is, in the observation plane within the circle near the axis with the radius of diffraction sector aperture and outside it. As a result of sufficiently labor-intensive mathematical procedures using the properties of incomplete cylindrical functions [4], which are not presented here, the following analytical expressions were obtained for Eq. [3].

Case 1: $n_0 \leq n$.

$$\begin{aligned}\Phi(\rho_0,\phi) = &-\frac{\exp\,[\,i\pi(n_0+n)]}{4}\cdot[\,V_0^-(\phi,p) - i\,V_1^-(\phi,p)\,] - \frac{i}{2}F(\upsilon_3)\cdot[F(\upsilon_2)-F(\upsilon_1)]+\\ &+\frac{\exp\,(i\pi\,n_0)}{4}\cdot\left[\sum_{k=0}^{\infty}(-1)^k\frac{(2\pi\,n_0)^{2k}}{A_{2k}}C_{2k} - i\sum_{k=0}^{\infty}(-1)^k\frac{(2\pi\,n_0)^{2k+1}}{A_{2k+1}}C_{2k+1}\right]\end{aligned} \tag{8}$$

Here, similar to Lommel problem at the circle [2] the functions are introduced, which have a formal resemblance with Lommel functions [6].

$$V_S^-(\phi,p) = \sum_{k=0}^{\infty}(-1)^k\left(\sqrt{\frac{n_0}{n}}\right)^{2k+S}\cdot E_{2k+S}^-(\phi,p), \tag{9}$$

where $E_\nu^-(\phi,p)$ are the incomplete cylindrical functions in Poisson form [4,5], which have an integral representation

$$E_\nu^-(\phi,p) = \frac{2p^\nu}{A_\nu}\int_0^\phi \exp\,(-ip\;\cos t\,)\sin^{2\nu} t\;dt \tag{10}$$

and the representation in the form of a power series relative to variable p

$$E_\nu^-(\phi,p) = \frac{p^\nu}{A_\nu}\sum_{k=0}^{\infty}C_{k,\nu}(\phi)\,\frac{(-i\,p)^k}{k\,!},\ \text{where}\quad C_{k,\nu}(\phi) = 2\int_0^\phi \cos^k t\,\sin^{2\nu} t\;dt$$

The rest quantities introduced to Eq. (8) are following: $F(\nu)$ is well-known Fresnel integral with parameters $\upsilon_1 = -\sqrt{2n_0}\cos\phi$, $\upsilon_2 = \upsilon_1 + \sqrt{2n}$, $\upsilon_3 = \sqrt{2n_0}\sin\phi$;

$C_\nu = 2\int_0^\phi \sin^{2\nu} t\;dt$; $A_\nu = 2^\nu\,\Gamma\,(\,\nu+\frac{1}{2}\,)\,\Gamma\,(\,\frac{1}{2}\,)$, where $\Gamma\,(\,\nu+\frac{1}{2}\,)$ is gamma-function with natural values ν; $p = 2\pi\,\sqrt{n_0}\,\sqrt{n}$ is a parameter.

All series including in Eq. (8) are absolutely convergent series, but they are convenient for computation when $n_0 \leq n$, i.e., in the observation plane within a circle near an axis with the radius of diffraction sector aperture. The obtained solution is true for arbitrary radius values and opening angle of the sector.

A special case confirming the validity of obtained solution is the diffraction problem by the circle aperture. In this case when $\varphi = 2\pi$ the incomplete cylindrical function in Poisson form

(10) become to Bessel functions $J_\nu(p)$ [6], introduced functions $V_S^-(\varphi, p)$ (9) become to Lommel functions V_S (n, n_0) [6], the second term in Eq. (8) vanishes owing to, and the third term becomes to unit, as the multiplier in squire brackets $\varphi = 2\pi$ is merely expansion into a series of function $\exp(-i\pi n_0)$. So, we obtain well-known Lommel solution [3]:

$$\Phi(\rho_0, 2\pi) = -\exp[i\pi(n_0 + n)]\,[\mathrm{U}_2(n, n_0) + i\,\mathrm{U}_1(n, n_0)]$$

where $V_S(n, n_0) = \sum\limits_{k=0}^{\infty}(-1)^k\left(\sqrt{\frac{n_0}{n}}\right)^{2k+S} J_{2k+S}(p)$ is Lommel function.

Case 2: $n_0 \geq n$.

$$\Phi(\rho_0, \phi) = -\frac{\exp[i\pi(n_0+n)]}{4}[U_2(\phi, p) + iU_1(\phi, p)] + \frac{\exp(i\pi n_0)}{2}\sin\phi\cos\phi\sum_{k=1}^{\infty}(-1)^k G_k \cdot J_{2k-1} \tag{11}$$

$$\text{where}\quad G_k = (iq)^{k+1}\frac{\sin^{2k}\phi}{A_{k+1}}, \tag{12}$$

$$J_k = \int_0^1 e^{i\frac{1}{2}qu^2}u^{2k+1}e^{-ipu\cos\phi}du \tag{13}$$

Here, similar to Lommel problem by circle [3], the functions are introduced, which have formal resemblance with Lommel functions.

$$U_S^-(\phi, p) = \sum_{k=0}^{\infty}(-1)^k\left(\sqrt{\frac{n}{n_0}}\right)^{2k+S}\cdot E_{2k+S}^-(\phi, p), \quad \text{where} \tag{14}$$

$U_\nu^-(\phi, p)$ are the incomplete cylindrical functions in Poisson form [4], which have an integral representation

$$E_\nu^-(\phi, p) = \frac{2p^\nu}{A_\nu}\int_0^\phi \exp\,(-ip\,\cos t)\sin^{2\nu}t\,dt \tag{15}$$

and the representation in the form of power series relative to variable p

$$E_\nu^-(\phi, p) = \frac{p^\nu}{A_\nu}\sum_{k-0}^{\infty}C_{k,\nu}(\phi)\frac{(-ip)^k}{k!}, \tag{16}$$

$$\text{where}\quad C_{k,\nu}(\phi) = 2\int_0^\phi \cos^k t\cdot\sin^{2\nu}t\,dt \tag{17}$$

The series including to Eq. (14), when $n_0 \geq n$ are completely converging, and for calculations of series (13) we present convenient recurrent formulas relating neighboring three terms of the series:

$$J_m = Q - (m-1)J_{m-2} + \sqrt{\frac{n_0}{n}}\cos\phi\cdot J_{m-1},$$

$$\text{where}\quad Q = e^{i\pi(n - \sqrt{n_0 n}\cos\phi)} \tag{18}$$

Thus, having initial values J_0 and J_1, one can calculate all subsequent values of $\mathrm{J_m}$..

As a whole, the solution in form (11) is convenient for calculations in the observation plane outside of the circle near the axis with the radius of sector aperture. The obtained solution is true for arbitrary values of radius and opening angle of the sector.

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A special case. The diffraction problem on the aperture of circular shape is one of the special cases confirming the validity of obtained solution. In this case when $\varphi = 2\pi$ incomplete cylindrical functions in Poisson form (15) become to Bessel functions [6], introduced function (14) become to Lommel function [6], and the second term in Eq. (11) vanishes. Thus, we obtain well-known Lommel solution [3]:

$$\Phi\,(\,\rho_0,\ 2\pi\,)\ =\ -\ \exp\ [\,i\,\pi\,(\,n_0 + n\,)\,]\,[\,\mathrm{U}_2(\,n, n_0) + i\,\mathrm{U}_1(\,n, n_0)\,],$$

where $U_S\,(\,n,\ n_0)\ =\ \sum\limits_{k=0}^{\infty}(-1\,)^k\left(\sqrt{\frac{n}{n_0}}\right)^{2k+S} J_{2k+S}(p)$ is Lommel function.

4. Calculation results

4.1. Spatial pattern of diffraction field from the sector aperture

Using the found solution, the patterns of the field spatial structure for sector apertures with arbitrary values of radius and opening angle were obtained. As an example, Fig, 2 shows the spatial structure of the field for sector aperture, which is symmetrically located relative to x axis and has the specified parameters: R = 10.95 λ and φ = 2π/3 (opening angle). The field structure is represented in the form of distribution Φ in three planes, distant from the aperture plane at distances $r_2\ =\ 180\,\lambda$, $r_2\ =\ 200\,\lambda$, $r_2\ =\ 230\,\lambda$.

The distance from the point source to the aperture plane is $r_1 = 300\ \lambda$. In each plane the values Φ, corresponding to a certain value of angular coordinate φ_0, are plotted radially from the center in units presented in Fig. 2. Each curve corresponds to specific value ρ_0 (i.e., the distance from the center). Distribution curves Φ have an asymmetry relative to axis y that is understandable as the aperture itself has the same asymmetry. All magnitudes of parameters and the proportions between them are selected for carrying out the model experiments when development of protective diffraction screens in microwave range.

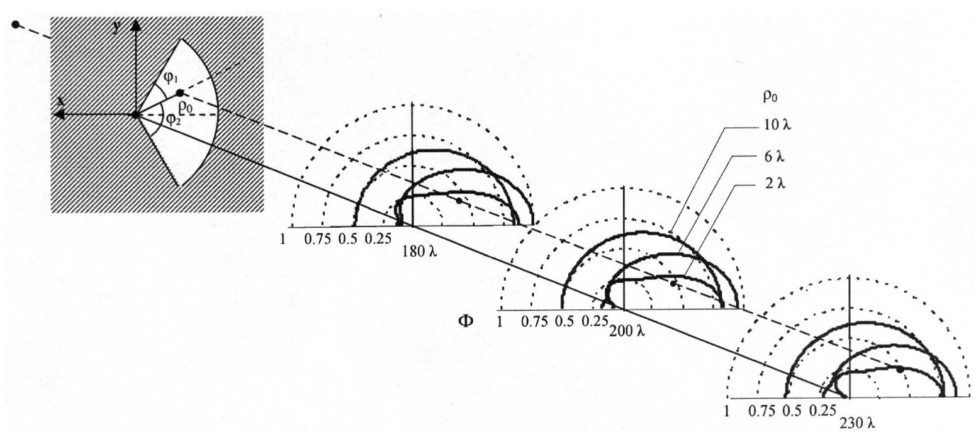

Fig. 2. Task geometry and the spatial structure of diffraction field (of the multiplier of diffraction attenuation Φ) for the sector aperture with radius R = 10,95 λ and opening angle φ = 2π/3. The distance from aperture to the source is $r_1 = 300\ \lambda$, and the distances to the observation planes are $r_2 = 180\ \lambda$, $r_2 = 200\ \lambda$, $r_2 = 230$. Each curve corresponds to a certain remoteness from the central axis: $\rho_0 = 10\ \lambda$, $\rho_0 = 6\ \lambda$, $\rho_0 = 2\ \lambda$.

4.2. Spatial pattern of the diffraction field from the attenuating screen

The screen in the form of a semicircle with the radius equal to the radius of the first Fresnel zone in the plane of the screen relative to the points of radiation and observation, i.e. $\mathrm{R} = b_1$.

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is of interest in the practice of microwave radio range. Such screen at the observation point at the central axis, passing through the semicircle center, gives the field attenuation up to zero. Naturally, the distribution of the field near these points is of interest. The results of calculations of diffraction pattern of the field near the point of its minimum level for the screen in the form of semicircle with radius R = 10.95λ are presented in Fig. 3. These results are obtained using Babine principle [1,7] and found solution (3) for the sector aperture.

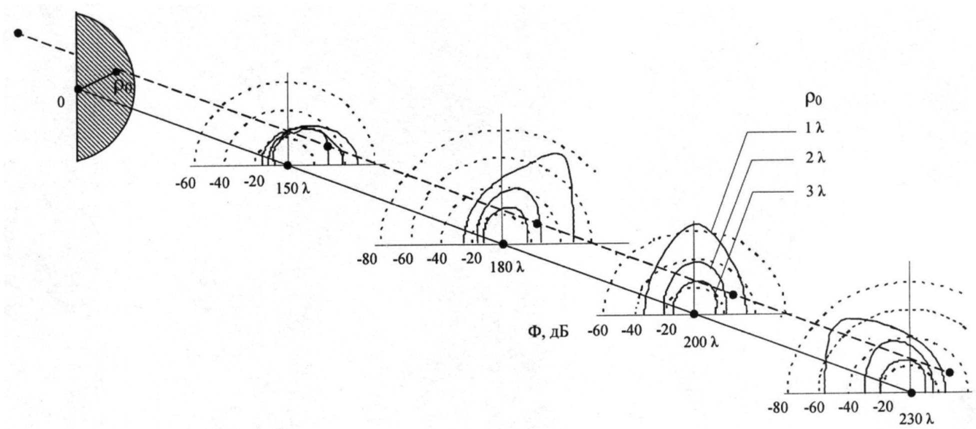

Fig. 3. Spatial field patterns (of multiplier of diffraction attenuation Φ) for attenuating screen in the form of semicircle with radius R. The distance from the aperture to the source is $r_1 = 300\lambda$, and the distances to the observation planes are $r_2 = 150\lambda$, $r_2 = 180\lambda$, $r_2 = 200\lambda$, $r_2 = 230\lambda$. Each curve corresponds to a certain distance from the central axis: $\rho_0 = \lambda$, $\rho_0 = 2\lambda$, $\rho_0 = 3\lambda$.

For the screen with radius R = 10.95λ the design point of the lowest level of the field is located at the central axis at distance from the screen plane, so the plane, in which this point lies is the main one. The field structure is presented in the form of spatial distribution pattern Φ in the main plane at distance. and in three other planes, distant from the aperture plane to distances $r_2 = 150\,\lambda$, $r_2 = 180\,\lambda$, $r_2 = 230\,\lambda$. In each plane the values of Φ, corresponding to a certain value of angular coordinates φ_0, are plotted radially from the center in the units presented in the figure. Since the screen is attenuating type, and the field levels reach very small values, the logarithmic units (dB) are selected here. Each curve corresponds to a certain distance of observation point ρ_0 from the center.

As it is seen from field diffraction patterns, the field attenuation degree decreases as moving off from the central point both along the axis, and in the observation plane itself. In the main plane ($r_2 = 200$ λ) at the small distance from center $\rho_0 =$ λ the attenuation reaches more than -60 dB, while at distance $\rho_0 = 3$λ there is an attenuation of about -20 dB. The same can be noted when shifting along the axis. The diffraction pattern has some asymmetry relative to vertical axis, which is amplified when removing off from the main plane of attenuation. At that the direction of asymmetry depends on the position of observation point relative to the main plane of attenuation.

5. Conclusion

1. By transformation of surface integral when Fresnel diffraction by sector aperture, the analytical expression for diffraction field in the form of series by incomplete cylindrical functions in Poisson form was obtained, which converge well that results to the construction of effective computational algorithms throughout the observation plane in Fresnel zone. As an example, the calculation results of the spatial structure of diffraction field from the sector aperture and from the protective screen in the form of semicircle are presented.

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2. Owing to a large community, the obtained problem solution can serve as a basis for construction of a whole range of solutions when the diffraction at other angular and segment areas where there are no analytical solutions.

Disclosures. The authors declare no conflicts of interest.

Data availability. No data were generated or analyzed in the presented research.

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