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ELECTRODYNAMICS AND WAVE PROPAGATION

The Transmission and Reflection Coefficients of an Electromagnetic Wave for a Gradient Dielectric Layer

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Abstract—A gradient dielectric layer is considered. In the layer, permittivity $\varepsilon(z)$ depends on coordinate z as follows: $\varepsilon(z) = \varepsilon_L a^4/(z-a)^4$ (a being the gradientness parameter). For this layer, reflection and transmission coefficients r_r and r_t are determined. It is found that, in a wide frequency range, the considered gradient layer has amplitude- and phase-constant reflection and transmission coefficients and can serve as a low-pass filter in radio devices involved in the transmission of wideband signals.

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INTRODUCTION

For radio devices radiating, transmitting, and receiving wideband electromagnetic signals, media with gradient electric parameters—conductance σ and permittity ε—can be useful [1, 2]. Such media can be created through spraying molecular beams on a substrate, chemical growing, and packing thin plates in a file where the conductance and permittivity of each plate differ from those of the other plates [3]. For simulating and analyzing the operation of such devices containing a gradient layer as an element, it is important to have, as a model, a problem that can be solved exactly. As a model, we consider a layer with a gradient permittivity that admits the exact calculation of the electromagnetic field components in the layer and the reflection and transmission coefficients of this layer. Usually, a general approach allowing the determination of exact solutions is the method of reduction (performed with the help of replacement of variables and functions) [4] of the Helmholtz equation to a certain reference differential equation having known solutions [2]. Alternative reduction methods [5, 6] substantially differing from the methods described in [4] have recently been proposed. These new methods are applied to solve the Maxwell equations and allow obtaining solutions differing from those presented in [2].

In the study, we consider a general solution to the problem for a gradient medium. The obtained results are verified with the use of the limit passage from a gradient to homogeneous medium. In addition, a semi-infinite gradient medium is considered in the case when a plane electromagnetic wave is normally incident on the medium in a homogeneous dielectric. For this case, the reflection and transmission coefficients are found. A solution is obtained for a finite-thickness gradient dielectric layer.

The problem under study has recently become topical. Thus, in [7], the idea of reconstructing the electric parameters of a reflecting layer from the envelope of the frequency characteristic of the reflection coefficient is used. As in our study, in [8], an inhomogeneous layer is considered. However, the authors of [8] tried to find a solution in the form of a series in the spatial coordinate, whereas we consider the formulation of the problem that admits an exact analytic solution for both the electromagnetic field components and the reflection and transmission coefficients.

1. A GENERAL SOLUTION

Assume that the electromagnetic wave depends on time as $\exp(-i\omega t)$, where ω is the circular frequency. Below, time derivatives are not used, and, therefore, the electromagnetic field components do not contain the exponential factor $\exp(-i\omega t)$. In an inhomogeneous dielectric medium, the wave equation for the tangential component of electric field E has the form

$$E'' + \frac{\omega^2}{c^2}\varepsilon(z)E = 0, \tag{1}$$

where c is the velocity of light, two primes denote the double differentiation with respect to z, and coordinate z is measured along the direction from the interface into the gradient medium. Following studies [5, 6], we seek for a solution to Eq. (1) in the form

$$E = \frac{1}{\sqrt{\varphi'(z)}}\exp\left(i\frac{\omega}{c}\varphi(z)\right). \tag{2}$$

This technique substantially differs from the reduction method considered in [2]. The substitution of (2) into (1) yields

$$\varepsilon(z) = \varphi'^2 - \frac{c^2}{\omega^2}\left(\frac{3}{4}\frac{\varphi''^2}{\varphi'^2} - \frac{\varphi'''}{2\varphi'}\right). \quad (3)$$

Here, we have one equation for two functions $\varepsilon(z)$ and $\varphi(z)$. However, if the bracketed expression is set zero, we obtain two necessary equations from (3). One of these

$$\frac{3}{4}\frac{\varphi''^2}{\varphi'^2} - \frac{\varphi'''}{2\varphi'} = 0$$

is elementarily integrated:

$$\varphi = \frac{C_1}{C_2 + z} + C_3, \quad (4)$$

where $C_{1,2,3}$ are integration constants. From the other equation $\varepsilon(z) = \varphi'^2$, we find

$$\varepsilon(z) = \frac{C_1^2}{(C_2 + z)^4}.$$

A suitable representation of the solution for the permittivity is as follows:

$$\varepsilon(z) = \frac{\varepsilon_L a^4}{(z - a)^4}. \quad (5)$$

The boundary value of the permittivity is $\varepsilon = \varepsilon_L$ on the upper boundary at $z = 0$ and $\varepsilon(h) = \varepsilon_L a^4/(a - h)^4$ on the lower boundary at $z = h$. Quantity a, which is measured in length units, can be referred to as the gradientness parameter. Depending on the required conditions imposed on the gradient layer, gradientness parameter a can be negative as well. For expression (5) to be convergent, it is necessary that quantity a is not equal to thickness of h of the gradient layer. The transition layer becomes homogeneous with a constant value of permittivity ε_L if we pass to the limit as $a \to \infty$.

The substitution of (4) into expression (2) yields the solution for the electric field

$$E = \frac{a - z}{a}\text{const}\exp\left(\pm ik_L\frac{a^2}{a - z} \mp ik_L a\right). \quad (6)$$

Here, the parameter

$$k_L = \frac{\omega}{c}\sqrt{\varepsilon_L} \quad (7)$$

is introduced and signs "±" refer to the incident and reflected waves, respectively. The additional phase that enters the exponent and is independent of z is due to the condition that, upon passing to the inhomogeneous medium, the electric field is determined by the relationship

$$E = \text{const}\exp(\pm ik_L z), \quad (8)$$

which is a solution to the wave equation with constant coefficients. Actually, in the limit $a \to \infty$, the exponent from solution (6) takes the form

$$\pm ik_L\frac{a^2}{a - z} \mp ik_L a$$
$$\approx \pm ik_L\left(a + z + \frac{z^2}{a} + \dots\right) \mp ik_L a = \pm ik_L z,$$

and expression (6) becomes (8).

In addition, we present the expression for the electric field in the case of a unit-amplitude incident wave:

$$E_1 = \exp(ik_1 z) + A\exp(-ik_1 z). \quad (9)$$

Here, factor A is the reflection amplitude and the wave number in a homogeneous dielectric medium is calculated from the equation

$$k_1 = \frac{\omega}{c}\sqrt{\varepsilon_1}, \quad (10)$$

where ε_1 is the permittivity of the homogeneous medium. The reflected wave (the term with amplitude A in formula (9)) is due to the presence of the interface between the homogeneous and gradient media. Let us comment on the foregoing in more detail. The permittivity of an inhomogeneous medium has a gradient. One of the techniques of approximating such a medium involves its modeling with thin homogeneous layers with uniform values of the permittivity in each layer. For this model, a linear coupling between the electromagnetic field components is retained. When the inhomogeneous structure of a medium is modeled with thin homogeneous layers, the Maxwell equations remain linear. In each layer, Maxwell equations have solutions in the form of transmitted and reflected waves. Therefore, the concept of a reflected wave can be introduced for a gradient or an inhomogeneous medium. Then, the model of a gradient medium is more general than the model of a homogeneous-layer medium. The gradient dependence of the permittivity makes it possible to consider media with wideband frequency properties.

The tangential magnetic induction component orthogonal to the electric field can be found from the expression

$$B = -\frac{i}{\omega}E', \quad (11)$$

which follows from the Maxwell equations and is identical in all media with different spatial dependences of the permittivity. The substitution of (9) into (11) yields the magnetic induction of the incident wave:

$$B_1 = \frac{k_1}{\omega}\exp(ik_1 z) - \frac{k_1}{\omega}A\exp(-ik_1 z). \quad (12)$$

2. A SEMI-INFINITE GRADIENT DIELECTRIC MEDIUM

According to solution (6), wave E transmitted into the gradient medium has the form

$$E = \frac{a-z}{a} R \exp\left(ik_{\mathrm{L}} \frac{a^2}{a-z} - ik_{\mathrm{L}} a \right). \quad (13)$$

The choice of signs in the exponent is determined by the condition that, in the limit $a \to \infty$, we obtain the transmitted plane wave $E = R\exp(ik_{\mathrm{L}} z)$, where R is the transmission amplitude.

The substitution of function (13) into (11) yields the expression for the magnetic induction

$$B = \frac{R}{a\omega}\left(i + \frac{k_{\mathrm{L}} a^2}{a-z} \right) \exp\left(ik_{\mathrm{L}} \frac{a^2}{a-z} - ik_{\mathrm{L}} a \right). \quad (14)$$

Only the nonzero components are presented for the electric field and magnetic induction.

The boundary conditions mean that, at $z = 0$, components E_1 and E, as well as B_1 and B, are equal, whence we obtain two equations for A and R. Solving these equations, we find

$$A = \frac{k_1 - k_{\mathrm{L}} - i/a}{k_1 + k_{\mathrm{L}} + i/a}, \quad (15)$$

$$R = \frac{2k_1}{k_1 + k_{\mathrm{L}} + i/a}. \quad (16)$$

In the limit $a \to \infty$, we can neglect i/a, whence the known formulas expressing the reflection and transmission coefficients for a homogeneous medium follow [1]. Using the time dependence of the fields in the form $\exp(-i\omega t)$, we deal with complex quantities. Physical quantities are the real parts of the corresponding complex quantities.

Let calculate reflection and transmission coefficients r_{r} and r_{t}. To this end, we present the expression $\vec{S} = c\vec{E} \times \vec{B}$ for vector $\vec{S}$, which we call the electromagnetic wave flux. Accurate to a factor that is the same in any media, $\vec{S}$ is the Poynting vector [1]. In the chosen coordinate frame, we have the only nonzero component $S = cEB$. This expression should be averaged over time: $S = \langle cEB \rangle$. In addition, since we deal with complex quantities, S should be replaced by the expression [9]

$$S = \frac{c}{2}\langle EB^* + E^*B \rangle. \quad (17)$$

The asterisk denotes complex conjugation. The substitution of functions (9) and (12) into (17) yields the incident wave flux

$$S_1 = \sqrt{\varepsilon_1}, \quad (18)$$

and the reflected wave flux

$$S_A = \sqrt{\varepsilon_1}|A|^2. \quad (19)$$

Similarly, using (13) and (14), we find the transmitted wave flux

$$S_R = \sqrt{\varepsilon_{\mathrm{L}}}|R|^2. \quad (20)$$

Now, according to a standard procedure, we find reflection coefficient r_{r}

$$r_{\mathrm{r}} = \frac{S_A}{S_1} = |A|^2 \quad (21)$$

or

$$r_{\mathrm{o}} = \frac{(k_1 - k_{\mathrm{L}})^2 + 1/a^2}{(k_1 + k_{\mathrm{L}})^2 + 1/a^2} = \frac{\left(\sqrt{\varepsilon_1} - \sqrt{\varepsilon_{\mathrm{L}}}\right)^2 + c^2/\omega^2 a^2}{\left(\sqrt{\varepsilon_1} + \sqrt{\varepsilon_{\mathrm{L}}}\right)^2 + c^2/\omega^2 a^2}$$

and transmission coefficient r_{t}

$$r_{\mathrm{t}} = \frac{S_R}{S_1} = \sqrt{\frac{\varepsilon_{\mathrm{L}}}{\varepsilon_1}}|R|^2 \quad (22)$$

or

$$r_{\mathrm{t}} = \sqrt{\frac{\varepsilon_{\mathrm{L}}}{\varepsilon_1}} \frac{4k_1^2}{(k_1 + k_{\mathrm{L}})^2 + 1/a^2} = \frac{4\sqrt{\varepsilon_1 \varepsilon_{\mathrm{L}}}}{\left(\sqrt{\varepsilon_1} + \sqrt{\varepsilon_{\mathrm{L}}}\right)^2 + c^2/\omega^2 a^2}.$$

It follows from (21) and (22) that $r_{\mathrm{r}} + r_{\mathrm{t}} = 1$, as it should be.

3. A GRADIENT DIELECTRIC MEDIUM OF A FINITE THICKNESS

Let a plane gradient medium be a layer that has thickness h and separates two homogeneous media with permitivities ε_1 and ε_2 (Fig. 2). Assume that the wave is normally incident from the medium with ε_1 and partly transmitted through the gradient layer into the medium with ε_2. We assume for definiteness that $\varepsilon_1 < \varepsilon_2$. For the opposite case $\varepsilon_1 > \varepsilon_2$, it suffices to reverse the sign of gradientness parameter a in formula (5). When artificial materials with given values of permittivities are created, gradientness parameter a can always be selected such that it does not approach the value of layer thickness h. This condition ensures the absence of the divergence of the expression $1/(a - h)$.

The electric field in a gradient medium has the form

$$\begin{aligned} E &= \frac{a-z}{a} U \exp\left(ik_{\mathrm{L}} \frac{a^2}{a-z} - ik_{\mathrm{L}} a \right) \\ &+ \frac{a-z}{a} W \exp\left(-ik_{\mathrm{L}} \frac{a^2}{a-z} + ik_{\mathrm{L}} a \right), \end{aligned} \quad (23)$$

where U is the amplitude of the electric field propagating in the layer, W is the amplitude of the electric field reflected by the lower boundary of the layer. In the limit $a \to \infty$, we obtain the expression for the field in a homogeneous layer medium [1]. Reflection and transmission amplitudes A and R, respectively, retain their physical meanings but are determined by expressions

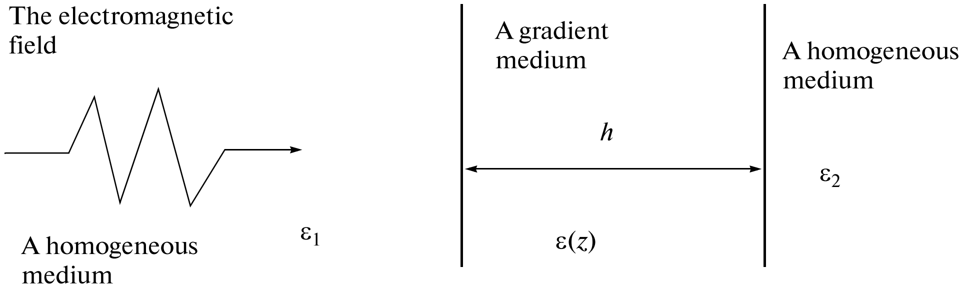

Fig. 1. Finite-thickness gradient layer illuminated from the left by a normally incident electromagnetic wave.

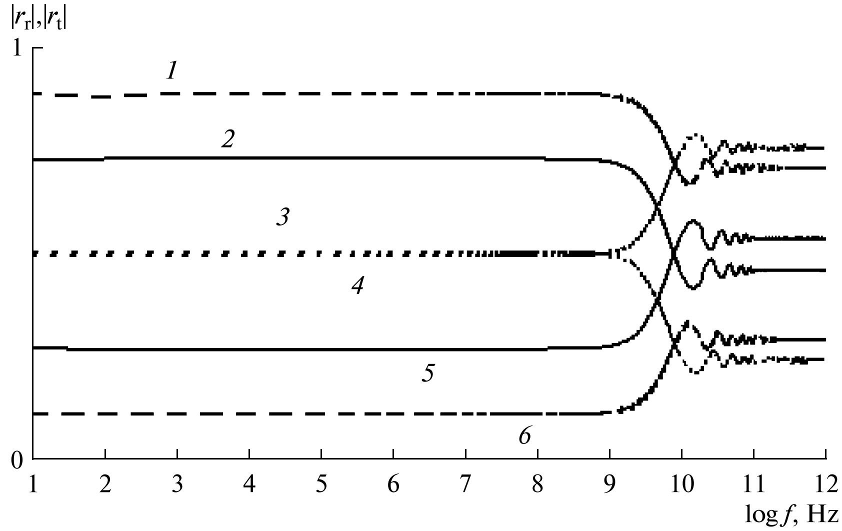

Fig. 2. Frequency characteristics of absolute values of the transmission and reflection coefficients (*1–3*) $|r_t|$ and (*4–6*) $|r_r|$ calculated at $a =$ (solid curve) 0.023, (dashed curve) 0.033, and (dotted curve) 0.017 m.

different from (15) and (16). From (11), we find the magnetic induction in the layer:

$$B = \frac{1}{\omega a}\left(i + \frac{k_L a^2}{a-z}\right) U \exp\left(ik_L \frac{a^2}{a-z} - ik_L a\right) + \frac{1}{\omega a}\left(i - \frac{k_L a^2}{a-z}\right) W \exp\left(-ik_L \frac{a^2}{a-z} + ik_L a\right). \quad (24)$$

For electric field E_2 transmitted through the layer into the second dielectric medium, we obtain

$$E_2 = R\exp(ik_2 z - ik_2 h), \quad (25)$$

where $k_2 = \frac{\omega}{c}\sqrt{\varepsilon_2}$. The magnetic induction in the second homogeneous medium is calculated from formula (11):

$$B_2 = \frac{k_2}{\omega} R\exp(ik_2 z - ik_2 h). \quad (26)$$

The boundary conditions on the upper and lower boundaries of the gradient layer yield a system of four equations for amplitudes A, U, W, and R:

$$1 + A = U + W,$$

$$k_1 a(1 - A) = (i + k_L a)U + (i - k_L a)W,$$

$$R = \frac{a-h}{a} U \exp\left(\frac{ik_L ah}{a-h}\right) + \frac{a-h}{a} W \exp\left(-\frac{ik_L ah}{a-h}\right),$$

$$k_2 a R = \left(i + \frac{k_L a^2}{a-h}\right) U \exp\left(\frac{ik_L ah}{a-h}\right)$$

$$\times \left(i - \frac{k_L a^2}{a-h}\right) W \exp\left(-\frac{ik_L ah}{a-h}\right).$$

Let us introduce the parameters

$$a_1 = \frac{i + k_L a}{k_1 a}, \quad a_2 = \frac{i - k_L a}{k_1 a}, \quad (27)$$

$$n = \exp\left(\frac{ik_L ah}{a-h}\right), \quad (28)$$

$$\Delta = (1 + a_1)(b_2 - d_2) - (1 + a_2)(b_1 - d_1),$$

$$b_1 = \frac{a-h}{a} n, \quad b_2 = \frac{a-h}{a}\frac{1}{n}, \quad (29)$$

$$d_1 = \frac{i + \frac{k_L a^2}{a-h}}{k_2 a} n, \quad d_2 = \frac{i - \frac{k_L a^2}{a-h}}{k_2 a}\frac{1}{n}. \quad (30)$$

Then, a solution to the system of equations for the amplitudes is as follows:

$$A = \frac{(1-a_1)(b_2-d_2)-(1-a_2)(b_1-d_1)}{\Delta}, \quad (31)$$

$$V = \frac{2(b_2-d_2)}{\Delta}, \quad W = -\frac{2(b_1-d_1)}{\Delta}, \quad (32)$$

$$R = 2\frac{-b_1d_2+b_2d_1}{\Delta} = \frac{4k_L}{k_2\Delta}. \quad (33)$$

The above representation of the solution for the amplitudes seems to be the most suitable for numerical computation. It is hardly convenient to write cumbersome expressions for the amplitudes in an explicit form. Thus, for example,

$$\Delta = \left(1+\frac{1+k_La}{k_1a}\right)\left(\frac{a-h}{a}-\frac{1-\frac{k_La^2}{a-h}}{k_2a}\right)\exp\left(-i\frac{k_Lah}{a-h}\right)-\left(1+\frac{1-k_La}{k_1a}\right)\left(\frac{a-h}{a}-\frac{1+\frac{k_La^2}{a-h}}{k_2a}\right)\exp\left(i\frac{k_Lah}{a-h}\right).$$

The substitution of expressions (9) and (12) into (17) yields formula (21) for the reflection coefficient:

$$r_r = |A|^2,$$

where amplitude A is specified by expression (31). Similarly, the substitution of expressions (25) and (26) into (17) yields the following formula for the transmission coefficient:

$$r_t = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}|R|^2, \quad (34)$$

where amplitude R is specified by expression (33). One can show that, in the limit $a \to \infty$, we obtain the result from [10] for the reflection and transmission coefficients in the case $\varepsilon_1 = 1$ and $\varepsilon_2 = 1$. Thus, for example, we obtain in this case for amplitudes A and R

$$A = \frac{(1-\varepsilon_L)\exp\left(-i\omega h\sqrt{\varepsilon_L}/c\right)-(1-\varepsilon_L)\exp\left(i\omega h\sqrt{\varepsilon_L}/c\right)}{\left(1+\sqrt{\varepsilon_L}\right)^2\exp\left(-i\omega h\sqrt{\varepsilon_L}/c\right)-\left(1-\sqrt{\varepsilon_L}\right)^2\exp\left(i\omega h\sqrt{\varepsilon_L}/c\right)},$$

$$R = \frac{4\sqrt{\varepsilon_L}}{\left(1+\sqrt{\varepsilon_L}\right)^2\exp\left(-i\omega h\sqrt{\varepsilon_L}/c\right)-\left(1-\sqrt{\varepsilon_L}\right)^2\exp\left(i\omega h\sqrt{\varepsilon_L}/c\right)}.$$

In addition, one can check that, after some algebra, the equality

$$r_r + r_t = 1 \quad (35)$$

is obtained.

Figures 2 and 3 show the frequency characteristics of the absolute values of the reflection and transmission coefficients and the corresponding phases. These dependences are calculated with the use of formulas (27)–(34) for the following parameters:

$$\varepsilon_1 = 1, \ \ \varepsilon_L = 1, \ \ \varepsilon_2 = 10, \ \ h = 0.01 \text{ m},$$

for the reflection (curve *2*) and transmission (*5*) coefficients at $a = 0.023$ m (Fig. 2);

$$\varepsilon_1 = 1, \ \ \varepsilon_L = 1, \ \ \varepsilon_2 = 40, \ \ h = 0.01 \text{ m},$$

for the reflection (curve *1*) and transmission (*6*) coefficients at $a = 0.033$ m (Fig. 2); and

$$\varepsilon_1 = 1, \ \ \varepsilon_L = 1, \ \ \varepsilon_2 = 33, \ \ h = 0.01 \text{ m}, \ \ a = 0.017 \text{ m}$$

for the dotted lines (curves *3* and *4* in Fig. 2). The analysis of Figs. 2 and 3 yields the following results.

Within a wide frequency band of $1–10^9$ Hz (for the aforementioned parameters), the reflection and transmission coefficients (their absolute values and phases) are constant. As the gradientness parameter grows, the value of the transmission coefficient increases and, accordingly, the value of the reflection coefficient decreases. At the same time, the phases of the reflection and transmission coefficients are practically constant. This behavior indicates that, within a wide frequency band, the considered gradient layer can serve as a low-pass filter and a phase inverter in radio devices used for wideband signals. Note in addition that, with the appropriate value of gradientness parameter a, layers dividing the electromagnetic field by a given number can be created. For example, one half wave is transmitted through the layer and the other half wave is reflected by the layer (the dotted line in Fig. 2).

Starting from the frequency $f \approx 10^9$ Hz, the reflection and transmission coefficients (their absolute values and phases) begin to intensely oscillate. This effect is related to the fact that, at this frequency, the wavelength is comparable with the thickness of the transition layer. Actually, if c is the velocity of light, then, at the frequency $f \approx 10^9$ Hz, the wavelength is equal to the value

$$\lambda = c/f \approx 10^{-2} \text{ m},$$

which coincides with $h = 0.01$ m.

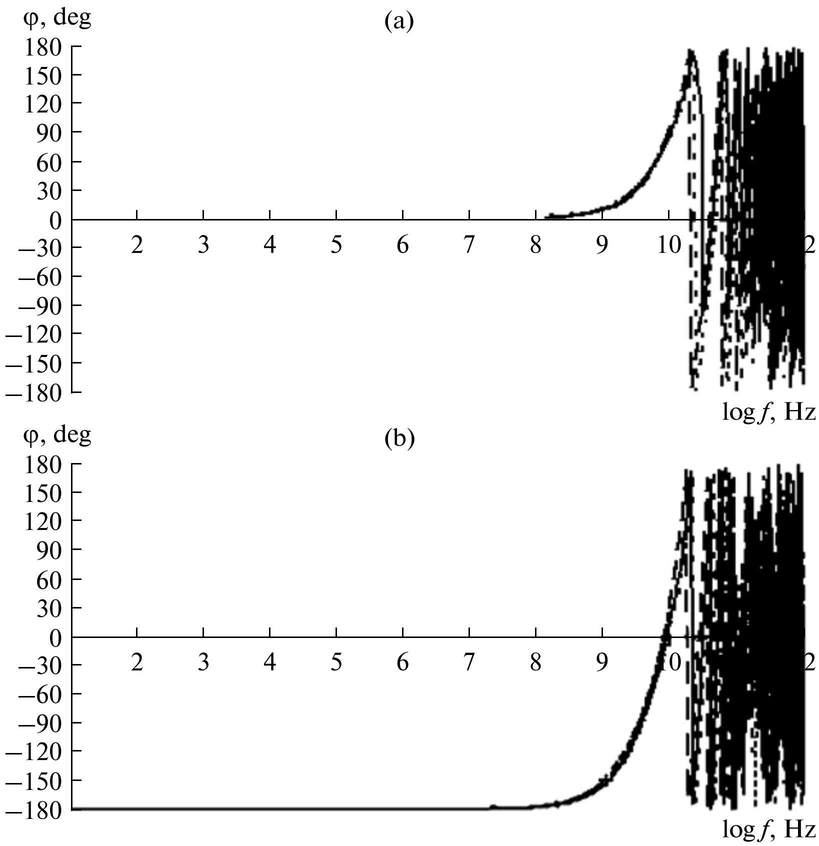

Fig. 3. Frequency characteristics of the phases of (a) transmission and (b) reflection coefficients obtained for various values of the gradientness parameter. The notation of curves coincides with that from Fig. 2.

CONCLUSIONS

The coefficients of reflection and transmission of a plane electromagnetic wave by a plane-parallel gradient dielectric layer have been calculated. A model with a gradient permittivity dependence that makes it possible to obtain results in a finite analytic form has been considered. For a gradient layer, the gradientness parameter has been introduced. Variation of this parameter makes it possible to change the passband and the values of the reflection and transmission coefficients. Such electrodynamic structure of the transition layer with specially selected electric parameters and thickness can serve as the matching element of a low-pass filter and as a phase inverter in radio devices applied for transmission of wideband and ultrawideband signals.

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REFERENCES

1. V. A. Neganov, E. I. Nefedov, and G. P. Yarovoi, *Electrodynamic Methods for Designing Microwave Devices and Antennas*, Ed. by V. A. Neganov (Radio i Svyaz', Moscow, 2002) [in Russian].
2. L. M. Brekhovskikh and O. A. Godin, *Acoustics of Layered Media*, Vol. 1: *Fundamentals of Reflection and Propagation of Sound* (Nauka, Moscow, 1989) [in Russian].
3. V. V. Gavrilin, Yu. I. Grigulis, and V. M. Porin'sh, *Electromagnetic Radiowave Device for Control of Layers of Semiconductor and Metallic Structures* (Zinatie, Riga, 1982) [in Russian].
4. E. Kamke, *Gewöhnliche Differentialgleichungen* (Academie, Leipzig, 1959; Nauka, Moscow, 1976).
5. A. G. Gantimurov and Yu. B. Bashkuev, J. Commun. Techn. Electron. **53**, 792 (2008).
6. A. B. Shvartsburg, Usp. Fiz. Nauk **170**, 1297 (2000).
7. O. V. Yushkova, J. Commun. Techn. Electron. **55**, 35 (2010).
8. M. V. Davidovich, J. Commun. Techn. Electron. **55**, 20 (2010).
9. L. D. Landau and E. M. Livshits, *Electrodynamics of Continuous Media* (Nauka, Moscow, 1982; Pergamon, Oxford, 1984).
10. I. V. Antonets, L. N. Kotov, V. G. Shavrov, and V. I. Shcheglov, J. Commun. Techn. Electron. **54**, 493 (2009).